A211869: Gauss taught me that.

R. J. Cano*†

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Abstract

Briefly it is presented here once more again an anecdotical fact from the life of the Prince of the Mathematics Karl Friedrich Gauss, as the preamble for the incidental implications of such fact inside the Algebra of matrices.

What the kid Gauss did: The sum of the first naturals.

When the little kid Gauss was asked to sum all the first 100 naturals. Presumably without taking place the chance to elapse five minutes after such assignation that was done by the teacher of the class, Gauss gave him the correct answer: 5050.

Here is what the history tell us that he explained:

$$\begin{aligned}1+2+3+4+5+6+7+8+9+\cdots+100\\ 100+99+98+97+96+95+94+93+92+\cdots+1\\ ------------------------\\ \frac{1}{2}(101+101+101+101+\cdots+101)\\ \frac{1}{2}(100)(101)\\ 5050\end{aligned}$$

Nowadays such observation became the well known identity:

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

*Email: remy at ula.ve

But the reader might be intrigued, asking: What have to do such kind of calculations with A211869 ?

Well, first than all A211869 is about a special kind of palindromes. Look the number 5050: Let us assume that there exist an hypotethical convention that assigns the Greek letter ψ as the writen numeral for the 50th digit in ascending order under base 100 numeration. There, 5050 is expressed as the palindrome $\psi\psi$ in analogy to the number 55 under decimal numeration.

But it is just the tip of an iceberg. Let us review again the work of the kid Gauss, but more in detail for an smaller case: The summation of the first ten naturals. If we were allowed to invoke diagonal matrices, then we will have something like this... Again the Gauss recipe:

$$\begin{array}{r} 1+2+3+4+5+6+7+8+9+10 \\ 10+9+8+7+6+5+4+3+2+1 \\ -------------------------- \\ \frac{1}{2}\left(11+11+11+\cdots+11\right)=\frac{1}{2}\left(10\right)\left(11\right)=55 \end{array}$$

But this time notice this interesting property of the particular way in that Gauss wrote the summand rows: Let us define a reading operation called "Zig-Zag" for the numbers in those summand rows as follows: 0) The rows are read decomposing them in columns from left to right. 1) Zig means to read first the number placed at bottom in the second row. 2) Zag means to read first the number placed at top in the first row. 4) After both numbers in a same column are read, we look to the right for the next column. If there left a column to read and it contains the same pair of numbers or the next column is the last then we keep the mode, else we alternate it between Zig and Zag. 5) This algorithm is repeated until all the cloumns at right are read. 6) If the last column were read indentical to the first, its numbers are swapped to preserve the symmetry.

For instance: By applying the Zig Zag to the two "Gaussian rows"[1] shown previously and by concatenating what we read, the result is[2]:

$$z_* \binom{1}{10} z^* \binom{2}{9} z_* \binom{3}{8} z^* \binom{4}{7} z_* \binom{5}{6} z_* \binom{6}{5} z^* \binom{7}{4} z_* \binom{8}{3} z^* \binom{9}{2} z^* \binom{10}{1} =$$
$$1, 0, 1, 2, 9, 8, 3, 4, 7, 6, 5, 5, 6, 7, 4, 3, 8, 9, 2, 1, 0, 1$$

We might notice there that the result is different depending on if we started to read either in a "Zig" or in a "Zag".

In the present example we started to read in "Zag". We got a palindromic pattern that might be turned into palindromic number if it were represented at least in decimal[3]. Such number would not be a prime:

$$1012983476556743892101 = (11)\,(401)\,(2659)\,(6323)\,(13659153463)$$

But beyond the curiosity shown previously about the possibility of finding out algorithms for building palindromes from the Gaussian rows, we can imagine them as the traces of diagonal matrices when we have to sum all the numbers contained by them in the identity about the summation for the first naturals:

$$(1+2+3+\ldots+N)=\frac{1}{2}\left[Tr\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & n \end{pmatrix} + Tr\begin{pmatrix} n & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

And this expression might lead us to think in the question: What would happen if we take the product between these matrices?

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & n \end{pmatrix} \begin{pmatrix} n & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[1]Let us refer to them inside the present work with this denomination.

[2]Z_* means "Zig", and Z^* means "Zag"

[3]Because what would be the greatest digit in such number have assigned 9 as value making 10 the smallest base where one can represent it with a single digit.

The answer is: If we take such product and then multiply the result by a column vector with all its components equal to the unit, we get the base independent representation for A211869(n) in the form of a column vector. Additionally if we call $\mathbf{W}_n$ to such representation, then its j-th component is given by: $(1 \leq j \leq n)$

$$W_n\left(j\right) = j(n - j + 1)$$

And now, we will have for each term in A211869 a family of infinitely many possible repreresentations depending on the base or radix where the components of $\mathbf{W}_n$ are read from up to down (or left to right if we transpose). Let be r the base, then we have:

$$\begin{aligned}\mathbf{A211869}\left(\mathbf{n},\mathbf{r}\right) &= \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{W_n}\left(\mathbf{j}\right)\mathbf{r}^{(\mathbf{n-j})} \\ &= \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{j(n-j+1)r^{(n-j)}}\end{aligned}$$

Finally, it is easy to realize that not all the bases work for getting palindromes. The reason is that in order to get palindromes all the components in $\mathbf{W}_n$ should be representable with a single digit. Such requisite is satisfied for every base r greater than the greatest component in $\mathbf{W}_n$. But how can we determine it in each case?. An application of the differential calculus might be helpful in this matter assuming temporarily a real variable context in the expression for $W_n\left(j\right)$.

Then let us optimize $W_n\left(j\right)$,

$$\begin{aligned}\frac{d}{dj}\left[W_n\left(j\right)\right] &= 0 \\ \frac{d}{dj}\left[j(n-j+1)\right] &= 0 \\ n - j + 1 - j &= 0 \\ j &= \frac{n+1}{2}\end{aligned}$$

Now replacing $j = \frac{n+1}{2}$ in $W_n\left(j\right)$ we get: $\frac{(n+1)^2}{4}$. Since we need to apply such result inside a integer context it is natural to take integer part using the floor function: $\lfloor\frac{(n+1)^2}{4}\rfloor$, and it is also necessary to add a unit because the obtained solution is the greatest digit but not the base what we are looking for, then r should be defined as $r\left(n\right) = 1 + \lfloor\frac{(n+1)^2}{4}\rfloor$.

A necessary correction

A problem arises when connecting $A211869$ with $A215940$: The Offsets.

In a $A215940$ the first term $a(1)$ should be zero because by definition we would be subtracting there the same quantities. In order to fix it all what we need here is to take the full expression for $A211869$ and change its right hand side replacing n with $(n-1)$ and subtracting 1 to the lower limit in the summation symbol. This gives[4]:

$$\mathbf{A211869}\,(\mathbf{n}) = \sum_{\mathbf{j=1}}^{\mathbf{n-1}} \mathbf{j}(\mathbf{n}-\mathbf{j}) \left[\mathbf{1} + \lfloor \frac{\mathbf{n^2}}{\mathbf{4}} \rfloor\right]^{(\mathbf{n-1-j})}$$

Conclusion

$A211869$ is both enough justified and well defined as a separate sequence relative to the maximum terms in $A215940$ of the form $A215940\,(k!)$. Additionally it allows to conjecture a possible connection between the palindromes obtained by the products among the diagonal matrices shown in the present work and the Pascal triangle: A triangular array made from the values of those digits in the palindromes defined by this sequence might be a modification of the Pascal triangle.

[4]Notice that we don't need to sum explicitly nothing that is granted to be zero.